DIAS-STP-86-04.

The Wu-Yang factor and the non-Abelian Aharonov-Bohm experiment

P.A. Horváthy

School of Theoretical Physics
Dublin Institute for Advanced Studies
10 Burlington Road, Dublin 4
Ireland

Abstract

The scattering of nucleons on a non-Abelian flux is shown to depend, as predicted by Wu and Yang, on the non-integrable phase factor.

PACS numbers: 11.15.-q, 11.30.Jw

The cornerstone of the Wu-Yang approach to gauge theory [1] is the non-integrable phase factor

$$(1) \quad \Phi = P(\exp - \oint A_j dx^j).$$

In particular, the scattering of a nucleon beam on a non-Abelian flux line-the non-Abelian version of the experiment proposed by Aharonov and Bohm for testing the role of electromagnetic potentials [2] - should depend on (1). This prediction has been confirmed recently using non-commuting path integrals [3]. The point is that such complications can be avoided. To see this, let us describe the scattering by a two-component Schrödinger equation

$$(2) \quad i\frac{\partial}{\partial t}\begin{pmatrix}\psi\\ \varphi\end{pmatrix} = -\frac{1}{2m}(\partial_j - A_j)^2\begin{pmatrix}\psi\\ \varphi\end{pmatrix}$$

where $A = A_j dx^j$ is an $SU(2)$ gauge potential whose field strength, $F_{jk} = \partial_j A_k - \partial_k A_j + [A_j, A_k]$, vanishes everywhere, except at the origin. Nucleons are identified with $SU(2)$ doublets. The validity of (2) is restricted, just like in the electromagnetic case, to the plane with the origin excluded. The remaining region is non-simply connected allowing for non-trivial values of (1).

The simplest way of confirming the Wu-Yang predictions is by paraphrasing the argument of Byers and Yang [4]. Define in fact

$$(3) \quad g(x) = P(\exp \int_{x_0}^{x} A_j dx^j)$$

where the integration is along an arbitrary path from a reference point x_0 to x which does not cross the positive x axis. The non-Abelian version of Stokes' theorem [5] implies [6] then that the integration is path-independent so that (3) provides us with a well-defined function. g(x) satisfies furthermore the relation $\partial_j g = -A_j g$. The (along the positive x-axis singular) gauge transformation $A_j \to g^{-1} A_j g + g^{-1} \partial_j g$, $\Psi = g^{-1} \binom{\psi}{\varphi}$, brings (2) to a <u>free</u> form:

(4) $i \dfrac{\partial \Psi}{\partial t} = -\dfrac{1}{2m} \Delta \Psi$

The new wave function becomes however <u>double-valued</u>:in polar coordinates (r, ϑ), we have

(5) $\Psi(r,0) = \Phi \Psi(r, 2\pi)$.

Consequently, (4) admits identical solutions as long as the "boundary conditions" (5) are the same [7], i.e. for identical Wu-Yang factors. Q.E.D.

To get an explicit solution it is convenient to use another [6] (regular) gauge: since any SU(2) group element can be diagonalized, $\Phi = \mathrm{diag}(\exp(2\pi i\alpha), \exp(-2\pi i\alpha))$ in a suitable gauge, where the real parameter α can be chosen to satisfy $0 \leq \alpha < 1$ with no loss of generality. A gauge potential is hence

(6) $A = A_\vartheta d\vartheta = \mathrm{diag}(\alpha, -\alpha)$,

and (2) splits hence into two, uncoupled, electromagnetic Bohm-Aharonov equations with electromagnetic potentials $\pm(\alpha/e)\,d\vartheta$:

(7) $\dfrac{\partial}{\partial t} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = -\dfrac{1}{2m} \left(\partial_r^2 + \dfrac{1}{r} \partial_r + \dfrac{1}{r^2} \begin{bmatrix} \partial_\vartheta + i\alpha & 0 \\ 0 & \partial_\vartheta - i\alpha \end{bmatrix}^2 \right) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$

with <u>periodic</u> boundary conditions this time. (7) is solved hence by simple applications of the Abelian results

$$(8)\quad \begin{pmatrix} \Psi_1(x,t) \\ \Psi_2(x,t) \end{pmatrix} = \int d^2y \begin{pmatrix} K_\alpha(x,t|y,0) & 0 \\ 0 & K_{-\alpha}(x,t|y,0) \end{pmatrix} \begin{pmatrix} \Psi_1(y,0) \\ \Psi_2(y,0) \end{pmatrix}$$

where the integration is on the punctured plane. The electromagnetic BA propagator K_α can be expressed as a path integral. By the well-known result ([7] and references therein):

$$(9)\quad K_\alpha = C_\alpha \sum_{n=-\infty}^{\infty} \chi^n K_n^0 \quad , \quad \chi = \exp[2\pi i \alpha] \, ,$$

where the integer n labels the homotopy classes of those paths which wind n times around the origin, K_n^0 is the corresponding partial propagator defined by the free dynamics. Explicitly [8]

$$(10)\quad K_n^0(x,t|y,0) = \left(\frac{m}{it}\right) \exp\left[\frac{im}{2t}(r^2 + r'^2)\right] \cdot$$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left[ik(\theta' - \theta + 2n\pi) - |k|\frac{\pi}{2}\right] J_{|k|}(mrr'/t)$$

where $x = (r,\vartheta)$, $y = (r',\vartheta')$ and J_m is the Bessel function. C_α is an overall (unobservable) phase factor.

But $\mathrm{diag}(\chi, \chi^{-1})$ is just the Wu-Yang factor Φ, and hence the non-Abelian propagator $\tilde{K} = \mathrm{diag}(K_\alpha, K_{-\alpha})$ is simply

$$(11)\quad \tilde{K} = \tilde{c}_\alpha \sum_{n=-\infty}^{\infty} K_n^0 \, \Phi^n$$

confirming once more the predictions of Wu and Yang.

$\tilde{C}_\alpha = \mathrm{diag}(C_\alpha, C_{-\alpha})$ is again an unobservable SU(2)-valued phase factor. The explicit formula (9) allows also for rederiving the scattering-expression in [6]. Observe finally that, due to the covariant transformation property $\Phi \to g^{-1} \Phi g$, (11) is valid in <u>any</u> gauge.

Notice that it is the existence of the diagonal gauge (6) which makes it unnecessary to working with non-commuting path integrals. However, this is true over an arbitrary Riemann surface, not only for the punctured plane [9].

References

[1] Wu T.T. and Yang C.N., Phys. Rev. D12, 3845 (1975)

[2] Aharonov Y. and Bohm D., Phys. Rev 115, 485 (1959)

[3] Botelho L. and De Mello, Rio de Janeiro preprint CBPF - NF - 062185 (1985)

[4] Byers N. and Yang C.N., Phys. Rev. Lett. 7, 46 (1961)

[5] Goddard P. and Olive D., Rep. Progr. Phys. 41, 1357 (1978) see also Halpern M.B., Nucl. Phys. B139, 477 (1978)

[6] Horváthy P.A., Phys. Rev. D33, 407 (1986); see also Asorey M., J. Math. Phys. 22, 179 (1981)

[7] The importance of boundary conditions is explained in Schulman L.S., Techniques and applications of Path Integration, New York, Wiley (1981): see also Henneberger W.C., Phys. Rev. Lett. 52, 573 (1984).

[8] Morandi G. and Menossi E. Eur. J. Phys. 5, 49 (1984); Berry M.V., Eur. J. Phys. 1, 240 (1980)

[9] Friedrich Th and Haberman L., Comm. Math. Phys. 100, 231 (1985)